模拟卷

**Problem 1**

1. **(5pt)** State the Perron-Frobenius theorem, ensuring that your answer incorporates the following key concepts: matrices with all positive entries, eigenvalues with multiplicity, spectral radius, and eigenvectors.
2. **(5pt)** Let be a real column vector satisfying . Determine all eigenspaces of the matrix , where denotes the vector of all ones.
3. **(10pt)** If, in addition, , prove that there exist indices and such that .

**Problem 2**

**(10pt)** Let be a square matrix of size with entries in , satisfying the following conditions:

* has all diagonal entries equal to zero.
* whenever .

Prove that , where is the null space of .

**Problem 3**

**(20pt)** Find invertible matrix and Jordan canonical form such that

**Problem 4**

**(20pt)** Find the singular value decomposition (SVD) of the matrix

i.e., determine , and such that .

**Problem 5**

**(10pt)** Let be an arbitrary field, and let be a collection of matrices. Prove the following equality:

where and denote the null space and row space of , respectively.

**Problem 6** Let be a complex matrix. Define its numerical range as:

From calculus, it is known that if , then for any , .

Now, assume further that .

1. **(5pt)** Prove that there exists a vector such that .
2. **(10pt)** Prove that there exists a unitary matrix such that has all diagonal entries equal to zero.
3. **(5pt)** Prove that there exist complex matrices and such that .